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Path integral quantization of scalar fluctuations above a kink

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Abstract

We quantize scalar fluctuations in 1+1 dimensions above a classical background kink. The properties of the effective action for the corresponding classical field are studied with an exact functional method, alternative to exact Wilsonian renormalization, where the running parameter is a bare mass and the regulator of the quantum theory is fixed. We extend this approach, in an appendix, to a Yukawa interaction in higher dimensions.

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1. Introduction

In the context of higher dimensional field theories, topological defects have been used to explain the localization of matter on four-dimensional branes. If one considers a scalar field defined on a non-trivial vacuum, with the shape of a kink centered on the brane, it is well known that massless chiral fermions coupled to the scalar field are localized on the corresponding brane [1]. In addition, this localization procedure can be used to define chiral fermions on the lattice, by using a kink-like mass term in the extra dimension [2]. Similar non-trivial topological effects, including sphalerons in real-time simulations, also lead to the description of chiral fermions on the lattice [3].

Analytical arguments toward this localization process are given at a classical level, and we consider here the quantization of scalar fluctuations above the kink, in order to exhibit non-perturbative properties. This quantization is stable in 1+1 dimensions only, if we consider the scalar field alone [4], what we will do here, since the corresponding toy model exhibits the main features we are interested in. In order to explain more specifically our motivations, though, we set up, in appendix B, the first steps of the generalization to a d+1 dimensional

Yukawa model, where fermions interact with scalar fluctuations above the kink. The present treatment does not take into account the collective coordinate corresponding to the translation invariance of the kink [4], since, in the spirit of the above-mentioned fermion localization problem, we consider here quantum fluctuations above one specific kink only, centered on $z = 0$, and we do not quantize the whole scalar theory, which contains a degenerate family of kinks. This is done in different papers [5], using canonical quantization. In more than 1+1 dimension, stability of quantum fluctuations necessitates the presence of another field than the scalar, and, in this context, the BRST (Becchi–Rouet–Stora–Tyutin) quantization of the nonlinear O(3) model was studied in [6].

The method we use here is an alternative to exact Wilsonian renormalization [7], where, instead of having fixed bare parameters and a running cut-off, we keep a fixed cut-off and consider a running bare mass, in the spirit of 'functional Callan Symanzik equations' [8]. The non-perturbative feature of this method, together with the absence of a running cut-off, led to the derivation of a cut-off-independent dynamical mass generated in the framework of a Kaluza–Klein model [9]. This method was also used for the description of time-dependent bosonic string actions [10], where a world sheet cut-off needs to be avoided, and where this alternative approach leads to new results, by studying the evolution of the quantum theory with the amplitude of the string tension.

In the present work, the cut-off which regulates the evolution equation for the effective theory will not appear in the evolution of the dressed parameters, and the logarithmic divergences expected in 1+1 dimensions are absent from our flows in the bare mass. Indeed, these flows are obtained after a differentiation with respect to the bare mass, which is equivalent to inserting an additional propagator in the graphs and therefore has the effect of reducing their degree of divergence. The physical interpretation of the evolution of the quantum theory with a bare mass is to control the amplitude of quantum fluctuations: when the bare mass is large, quantum fluctuations are frozen and the system is almost classical. As the bare mass decreases, quantum fluctuations gradually appear in the system, which therefore becomes dressed. A review can be found in [11].

From a technical point of view, this method can be seen as a tool, used to investigate properties of the quantum theory: the evolution in the bare parameter leads to a functional partial differential equation, which is then split into a series of differential equations, involving the dressed parameters which describe the effective theory. The integration of these non-perturbative differential equations leads to the effective theory, which exhibits the quantum properties of the system.

Section 2 describes the scalar model we study here, and shows the derivation of the evolution equation for the quantum theory with the bare mass of the quantum field which fluctuates above a classical background kink. The evolution equation we arrive at technically looks like an exact Wilsonian renormalization equation, but is actually very different in essence, as explained above.

Section 3 derives the evolution of the dressed parameters defining the quantum system, and discusses different properties of quantum theory. We show there that no odd power of the classical field is present in the effective action, whereas a cubic interaction is present in the bare action. We also compare our results to one loop predictions, and give new relations on the dressed parameters, beyond one-loop, as a consequence of the ressumation provided by our evolution equations.

Finally, section 4 contains a general discussion on our results, based on symmetry properties of the quantum theory. Appendix A shows the derivation of the evolution equations and appendix B displays the first steps on how to generalize the method to a (d+1)-dimensional Yukawa model.

2. Model and evolution of the effective theory

The bare action in 1+1 dimensions is

$$S_0 = \int \mathrm{d}t\, \mathrm{d}z \left\{ \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - U_B(\Phi) \right\}, \tag{1}$$

where z is the space coordinate, and the bare potential $U_B(\Phi)$ implements a spontaneous symmetry breaking:

$$U_B(\Phi) = -\frac{m_0^2}{2}\Phi^2 + \frac{\lambda_0}{24}\Phi^4. \tag{2}$$

In 1+1 dimensions, the scalar field has mass dimension 0, which leads to an important renormalization property: all the powers of the field are (classically) relevant operators, and all the coupling constants have mass dimension 2. As a consequence, the bare potential (2) is not chosen on the basis of relevance/irrelevance of the interactions, but rather on the assumption of small amplitude of fluctuations above the kink. This assumption will prove to be valid which will be seen with the effective theory that is obtained.

The classical equation of motion for the field is

$$\partial_\mu \partial^\mu \Phi + U_B^{'}(\Phi) = 0, \tag{3}$$

where a prime denotes a derivative with respect to Φ. We concentrate on the kink solution of equation (3) which depends on z only and reads

$$\Phi_{\mathrm{bg}}(z) = m_0\sqrt{\frac{6}{\lambda_0}}\tanh(\zeta), \tag{4}$$

where the dimensionless coordinate ζ is defined as

$$\zeta = \frac{m_0 z}{\sqrt{2}}. \tag{5}$$

We then consider the quantum fluctuations $\tilde{\Phi}$ around Φ_{bg} and write

$$\Phi(t,z) = \Phi_{\mathrm{bg}}(z) + \tilde{\Phi}(t,z). \tag{6}$$

If we take into account the equation of motion (3), the action depending on the dynamical variable $\tilde{\Phi}$ is

$$S = \int \mathrm{d}t\, \mathrm{d}z \left\{ \frac{1}{2}\partial_\mu \tilde{\Phi}\partial^\mu \tilde{\Phi} - m_0^2\tilde{\Phi}^2 - \frac{\lambda_0}{24}\tilde{\Phi}^4 + \frac{3}{2}m_0^2[1-\tanh^2(\zeta)]\tilde{\Phi}^2 - m_0\sqrt{\frac{\lambda_0}{6}}\tanh(\zeta)\tilde{\Phi}^3 \right\}. \tag{7}$$

We are interested in studying the quantum theory on the kink background, and we will derive for this the evolution of the effective action with the bare mass m_0. We will therefore start with the following bare action:

$$S_\xi = \int \mathrm{d}t\, \mathrm{d}z \left\{ \frac{1}{2}\partial_\mu \tilde{\Phi}\partial^\mu \tilde{\Phi} - \xi m_0^2\tilde{\Phi}^2 - \frac{\lambda_0}{24}\tilde{\Phi}^4 + \frac{3}{2}m_0^2[1-\tanh^2(\zeta)]\tilde{\Phi}^2 - \frac{g_0}{6}\tanh(\zeta)\tilde{\Phi}^3 \right\}, \tag{8}$$

where the dimensionless parameter ξ controls the amplitude of the mass term $m_0^2\tilde{\Phi}^2$, and $g_0 = m_0\sqrt{6\lambda_0}$. We will show that it is possible to derive an exact evolution equation for the effective action with ξ. The corresponding flows describe the evolution from $\xi \gg 1$, where the mass term dominates the Lagrangian and the theory is almost classical, to the expected quantum theory, obtained for $\xi = 1$.

We now proceed to the quantization of the system, integrating over the dynamical field $\tilde{\Phi}$. The partition function is

$$\begin{aligned} Z_\xi &= \int \mathcal{D}[\tilde{\Phi}] \exp\left(\mathrm{i}S_\xi[\tilde{\Phi}] + \mathrm{i}\int \mathrm{d}t\,\mathrm{d}z\, j\tilde{\Phi}\right) \\ &= \exp(\mathrm{i}W_\xi[j]), \end{aligned} \tag{9}$$

where j is the source and W_ξ is the connected graphs generator functional. The functional derivative of the latter defines the classical field ϕ:

$$\frac{\delta W_\xi}{\delta j} = \langle\tilde{\Phi}\rangle_\xi = \phi_\xi \qquad \frac{\delta^2 W_\xi}{\delta j \delta j} = -\mathrm{i}\phi_\xi\phi_\xi + \mathrm{i}\langle\tilde{\Phi}\tilde{\Phi}\rangle_\xi,$$

where

$$\langle\cdots\rangle_\xi = \frac{1}{Z_\xi}\int \mathcal{D}[\tilde{\Phi}](\cdots)\exp\left(\mathrm{i}S_\xi + \mathrm{i}\int \mathrm{d}t\,\mathrm{d}z\, j\tilde{\Phi}\right). \tag{10}$$

The effective action Γ_ξ (the proper graphs generator functional) is defined as the Legendre transform of W_ξ: after inverting the relation $j \to \phi_\xi$ to $\phi \to j_\xi$, one writes

$$\Gamma_\xi = W_\xi - \int \mathrm{d}t\,\mathrm{d}z\, j_\xi\phi, \tag{11}$$

where the source j_ξ has now to be seen as a functional of ϕ, parametrized by ξ. The functional derivatives of Γ are then

$$\begin{aligned} \frac{\delta\Gamma_\xi}{\delta\phi} &= -j_\xi \\ \frac{\delta^2\Gamma_\xi}{\delta\phi\delta\phi} &= -\frac{\delta j_\xi}{\delta\phi} = -(\delta^2 W_\xi)^{-1}_{jj}. \end{aligned}$$

The evolution equation for W_ξ with the parameter ξ is

$$\begin{aligned} \dot{W}_\xi &= -m_0^2\int \mathrm{d}t\,\mathrm{d}z\,\langle\tilde{\Phi}^2\rangle \\ &= -m_0^2\int \mathrm{d}t\,\mathrm{d}z\,\phi^2 + \mathrm{i}m_0^2\,\mathrm{Tr}\left\{\frac{\delta^2 W_\xi}{\delta j\delta j}\right\}, \end{aligned} \tag{12}$$

where a dot over a letter represents a derivative with respect to ξ. For the evolution of the effective action Γ, one should remember that its independent variables are ξ, ϕ, such that

$$\dot{\Gamma}_\xi = \dot{W}_\xi + \int \mathrm{d}t\,\mathrm{d}z\frac{\delta W_\xi}{\delta j}\partial_\xi j - \int \mathrm{d}t\,\mathrm{d}z\,\partial_\xi j\phi = \dot{W}_\xi. \tag{13}$$

Using the previous results, we finally obtain

$$\dot{\Gamma}_\xi + m_0^2\int \mathrm{d}t\,\mathrm{d}z\,\phi^2 = -\mathrm{i}m_0^2\,\mathrm{Tr}\left\{\left(\frac{\delta^2\Gamma_\xi}{\delta\phi\delta\phi}\right)^{-1}\right\}. \tag{14}$$

We stress here that although the right-hand side of equation (14) has the structure of a one-loop correction, this evolution equation provides a ressumation of all order in $\hbar$, since the effective action appearing in the trace contains the dressed action and not the bare one. Equation (14) is therefore a self-consistent equation, in the spirit of a differential Schwinger–Dyson equation, and is thus non-perturbative.

In order to extract the evolution of the dressed parameters defining the quantum theory, though, we need to adopt an approximation scheme and we assume, in the framework of the gradient expansion, the local potential approximation for Γ, with the kinetic term frozen to its classical expression, such that

$$\Gamma_\xi = \int \mathrm{d}t\,\mathrm{d}z\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U_\xi(\phi) + [1-\tanh^2(\zeta)]V_\xi(\phi) - \tanh(\zeta)Y_\xi(\phi)\right\}, \tag{15}$$

where U_ξ, V_ξ, Y_ξ are dressed potentials which define the quantum theory living on the kink and depend on the parameter ξ. These potentials will be determined by plugging the ansatz (15) into the evolution equation (14), and they read, at the tree-level,

$$\begin{aligned} U^{\text{tree}}(\phi) &= \xi m_0^2\phi^2 + \frac{\lambda_0}{24}\phi^4 \\ V^{\text{tree}}(\phi) &= \frac{3}{2}m_0^2\phi^2 \\ Y^{\text{tree}}(\phi) &= \frac{g_0}{6}\phi^3. \end{aligned} \tag{16}$$

In order to respect the symmetries of the bare action, in what follows we consider even potentials U_ξ, V_ξ and an odd potential Y_ξ

3. Evolution of the dressed parameters

In order to derive the evolution of the dressed parameters, we have to compute the trace appearing in the evolution equation (14), for a given configuration ϕ. Because of the symmetry of the function $\tanh(\zeta)$, a constant configuration for ϕ is not appropriate, as in such a case the derivative $\dot{Y}_\xi$ does not appear on the left-hand side of equation (14). The appropriate choice here is the step-like configuration

$$\phi_{\text{step}} = \text{sign}(z)\phi_0, \tag{17}$$

where ϕ_0 is a constant. This configuration has a singular kinetic term, but the corresponding singularity is ξ-independent in the framework of the gradient expansion (15), and therefore has no influence on the evolution in ξ. With such a configuration, the left-hand side of the evolution equation (14) is

$$LT\left[m_0^2\phi_0^2 - \dot{U}_\xi(\phi_0) - \dot{Y}_\xi(\phi_0)\right] + \frac{T}{m_0}[2\dot{V}_\xi(\phi_0) + \ln 2\dot{Y}_\xi(\phi_0)], \tag{18}$$

where T is the length of the time dimension and L is the length of the space dimension. These lengths being independent, one can independently identify in equation (14) the terms proportional to T and the terms proportional to LT.

The second derivative of the effective action is, for the configuration (17),

$$\begin{aligned} \frac{\delta^2\Gamma_\xi}{\delta\phi_1\delta\phi_2} &= -\{\partial_\mu\partial^\mu + U_\xi''(\phi_0)\}\delta(t_1-t_2)\delta(z_1-z_2) \\ &\quad + \{[1-\tanh^2(\zeta)]V_\xi''(\phi_0) - |\tanh(\zeta)|Y_\xi''(\phi_0)\}\delta(t_1-t_2)\delta(z_1-z_2). \end{aligned} \tag{19}$$

We then need the Fourier transform of the functions $|\tanh(\zeta)|$ and $1-\tanh^2(\zeta)$, and we find in appendix A

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}kz}[1-\tanh^2(\zeta)] &\simeq 4\frac{m_0^2}{k^3}\sin\left(\frac{k}{m_0}\right) - 4\frac{m_0}{k^2}\cos\left(\frac{k}{m_0}\right) \\ \int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}kz}|\tanh(\zeta)| &\simeq 2\pi\delta(k) + 2\frac{m_0}{k^2}\left[\cos\left(\frac{k}{m_0}\right) - 1\right]. \end{aligned} \tag{20}$$

We are interested in the limit of a strongly localized topological defect, and therefore consider the first order in $1/m_0$ only, where the previous Fourier transforms are

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-ikz}[1-\tanh^2(\zeta)] &\simeq \frac{4}{3m_0}\\ \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-ikz}|\tanh(\zeta)| &\simeq 2\pi\delta(k) - \frac{1}{m_0}.\end{aligned} \tag{21}$$

The Fourier transform of the second functional derivative (19) is then

$$\begin{aligned}\frac{\delta^2\Gamma_\xi}{\delta\phi_1\delta\phi_2} \simeq{}& \left\{\omega_1^2 - k_1^2 - U_\xi''(\phi_0) - Y_\xi''(\phi_0)\right\}2\pi\delta(\omega_1+\omega_2)2\pi\delta(k_1+k_2)\\ &+ \frac{1}{3m_0}\{4V_\xi''(\phi_0) + 3Y_\xi''(\phi_0)\}2\pi\delta(\omega_1+\omega_2),\end{aligned} \tag{22}$$

where we observe that, since translation invariance is broken in the space dimension, there is no conservation of momentum k in this direction. In what follows, we give the main steps of the derivations only, and the details can be found in appendix A.

3.1. Evolution of the potentials

For the step-like configuration (17), we evaluate the inverse of the second derivative (22) using the expansion

$$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \cdots, \tag{23}$$

where A is proportional to $\delta(\omega_1+\omega_2)\delta(k_1+k_2)$ and is thus diagonal and B is proportional to $\delta(\omega_1+\omega_2)$ only and is thus off-diagonal in the space dimension. In the previous expansion, the small parameter is k/m_0, where k is a typical IR momentum. The identification of the terms proportional to LT in the trace of equation (14) then gives

$$\dot U_\xi(\phi_0) + \dot Y_\xi(\phi_0) = m_0^2\phi_0^2 + \frac{m_0^2}{4\pi}\ln\left(1 + \frac{\Lambda^2}{U_\xi''(\phi_0) + Y_\xi''(\phi_0)}\right), \tag{24}$$

where a prime denotes a derivative with respect to the constant configuration ϕ_0 and Λ is the UV cut-off. The latter will actually not appear in the evolution equations for the parameters, since the expansion of equation (24) in powers of ϕ_0 leads to a field-independent divergence. We then choose the potentials such that $U_\xi(0) = 0$, $Y_\xi(0) = 0$, and subtract the corresponding evolution equation from equation (24) to obtain, in the limit $\Lambda \to \infty$,

$$\dot U_\xi(\phi_0) + \dot Y_\xi(\phi_0) = m_0^2\phi_0^2 + \frac{m_0^2}{4\pi}\ln\left(\frac{U_\xi''(0) + Y_\xi''(0)}{U_\xi''(\phi_0) + Y_\xi''(\phi_0)}\right). \tag{25}$$

The cut-off does not appear in our evolution equation as a consequence of the derivative with respect to a bare mass term, whereas we could expect logarithmic divergences in a (1+1)-dimensional field theory. The projection of equation (25) on the subspace of even functions of ϕ_0 gives the evolution of U_ξ and its projection on the subspace of odd functions gives the evolution of Y_ξ.

The evolution equations obtained after identification of the terms proportional to T is

$$\dot V_\xi(\phi_0) + \frac{\ln 2}{2}\dot Y_\xi(\phi_0) = -\frac{m_0^2}{24\pi}\left(\frac{4V_\xi''(\phi_0) + 3Y_\xi''(\phi_0)}{U_\xi''(\phi_0) + Y_\xi''(\phi_0)} - \frac{4V_\xi''(0) + 3Y_\xi''(0)}{U_\xi''(0) + Y_\xi''(0)}\right), \tag{26}$$

where the constant term has been chosen so as to respect $V_\xi(0) = 0$. In this latter equation also, the projection on the subspace of even functions gives the evolution of V_ξ and its projection on the subspace of odd functions gives the evolution of Y_ξ.

As is clear from equations (25), (26), a consistent solution for the potentials can be found only if $Y_\xi = 0$: these two evolution equations cannot give identical evolutions for Y_ξ. As a consequence, no odd function of the field appears in the effective theory. This property will be discussed in the last section, where we show that it is a consequence of symmetries of the quantum theory.

Finally, the effective action is

$$\Gamma_\xi = \int \mathrm{d}t\,\mathrm{d}z \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U_\xi(\phi) + [1-\tanh^2(\zeta)]V_\xi(\phi) \right\}, \tag{27}$$

where the dressed potentials U_ξ and V_ξ satisfy the evolution equations

$$\dot{U}_\xi(\phi_0) = m_0^2\phi_0^2 + \frac{m_0^2}{4\pi}\ln\left(\frac{U_\xi''(0)}{U_\xi''(\phi_0)}\right) \qquad \dot{V}_\xi(\phi_0) = -\frac{m_0^2}{6\pi}\left(\frac{V_\xi''(\phi_0)}{U_\xi''(\phi_0)} - \frac{V_\xi''(0)}{U_\xi''(0)}\right). \tag{28}$$

We observe that, in the framework of the gradient expansion (15), the evolution equation for U_ξ is independent of V_ξ. A further step in the gradient expansion would consist in taking into account quantum fluctuations in the kinetic term and write a general operator of the form $Z_\xi(\phi)\partial_\mu\phi\partial^\mu\phi$ in the effective action. The function Z_ξ would then couple the evolution equations for U_ξ and V_ξ.

Finally, we note that taking into account additional terms in expansion (23) would not influence the evolution of the effective potential U_ξ, but would add corrections of higher orders in $1/m_0$ to the evolution of V_ξ.

3.2. Truncation of the dressed potentials

Quantum fluctuations generate all the powers of field in the dressed potentials U_ξ, V_ξ. As discussed already, no operator is irrelevant here, in the Wilsonian sense, and therefore in principle one should take into account all the powers of ϕ. But if we assume small quantum fluctuations, we then consider the following truncation of the dressed potentials:

$$U_\xi(\phi_0) = \frac{M^2}{2}\phi_0^2 + \frac{\lambda}{24}\phi_0^4 + \frac{\beta}{6!}\phi_0^6 \qquad V_\xi(\phi_0) = \frac{v_1}{2}\phi_0^2 + \frac{v_2}{24}\phi_0^4 + \frac{v_3}{6!}\phi_0^6, \tag{29}$$

where the parameters $M^2, \lambda, \beta, v_1, v_2, v_3$ depend on ξ. This truncation takes into account the interactions which appear in the bare theory as well as the lowest interaction $\left(\phi_0^6\right)$ generated by quantum fluctuations. An expansion in powers of ϕ_0 in the evolution equation (28) for U_ξ gives, after identification of the different powers,

$$\begin{aligned}
\text{order } \phi_0^2: &\quad M\dot{M} = m_0^2 - \frac{\lambda m_0^2}{8\pi M^2}\\
\text{order } \phi_0^4: &\quad \dot{\lambda} = \frac{3m_0^2}{4\pi M^2}\left(\frac{\lambda^2}{M^2} - \frac{\beta}{3}\right)\\
\text{order } \phi_0^6: &\quad \dot{\beta} = \frac{15\lambda m_0^2}{2\pi M^4}\left(\frac{\beta}{2} - \frac{\lambda^2}{M^2}\right).
\end{aligned} \tag{30}$$

The identification of the powers of ϕ_0 in the evolution equation (28) for V_ξ gives

$$\begin{aligned}
\text{order } \phi_0^2: &\quad \dot{v}_1 = \frac{m_0^2}{6\pi M^2}\left(\frac{\lambda v_1}{M^2} - v_2\right)\\
\text{order } \phi_0^4: &\quad \dot{v}_2 = -\frac{m_0^2}{\pi M^2}\left(\frac{\lambda^2 v_1}{M^4} - \frac{\beta v_1}{6M^2} - \frac{\lambda v_2}{M^2} + \frac{v_3}{6}\right)\\
\text{order } \phi_0^6: &\quad \dot{v}_3 = -\frac{15m_0^2}{\pi M^4}\left[\frac{\lambda v_1}{M^2}\left(\frac{\beta}{2} - \frac{\lambda^2}{M^2}\right) + v_2\left(\frac{\lambda^2}{M^2} - \frac{\beta}{6}\right) - \frac{\lambda v_3}{6}\right].
\end{aligned} \tag{31}$$

If one desires to obtain the evolution of the dressed parameters with ξ, it is possible to solve equations (30), (31) numerically, but we give in what follows approximate analytical solutions, which contain the essential properties of the quantum theory.

3.3. One-loop approximation

We study here the one-loop approximation of the non-perturbative evolution equations for M, λ, β. For this, we note that the right-hand side of equation (14) contains the quantum corrections, such that the one-loop approximation is obtained by replacing on the right-hand side the dressed parameters by the bare ones: $M \to \sqrt{2\xi}m_0$, $\lambda \to \lambda_0$ and $\beta \to 0$. We then obtain for the one-loop parameters $M^{(1)}, \lambda^{(1)}, \beta^{(1)}$

$$
\begin{aligned}
M^{(1)}\dot{M}^{(1)} &= m_0^2 - \frac{\lambda_0}{16\pi\xi} \\
\dot{\lambda}^{(1)} &= \frac{3\lambda_0^2}{16\pi\xi^2 m_0^2} \\
\dot{\beta}^{(1)} &= -\frac{15\lambda_0^3}{16\pi\xi^3 m_0^4}.
\end{aligned}
\tag{32}
$$

It is interesting to compare these results with usual Feynman diagrams, obtained from the bare theory

$$
\int \mathrm{d}t\,\mathrm{d}z \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \xi m_0^2\phi^2 - \frac{\lambda_0}{24}\phi^4 \right\}, \tag{33}
$$

i.e. the initial bare theory without the z-dependent quadratic and cubic terms.

The one-loop correction to the parameter M^2 is generated by the interaction ϕ^4 which is represented by the tadpole diagram

$$
\begin{aligned}
(M^{(1)})^2 - 2\xi m_0^2 &= \frac{\mathrm{i}\lambda_0}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{1}{p^2 - 2\xi m_0^2} \\
&= \frac{\lambda_0}{2}\frac{\Omega_2}{(2\pi)^2}\int_0^\Lambda \frac{q\,dq}{q^2 + 2\xi m_0^2} \\
&= \frac{\lambda_0}{8\pi}\ln\left(1 + \frac{\Lambda^2}{2\xi m_0^2}\right),
\end{aligned}
\tag{34}
$$

where the factor $1/2$ takes into account the symmetry factor of the graph. It can be checked that the derivative of the latter result with respect to ξ indeed gives the expected expression (32) for $\partial_\xi\left[(M^{(1)})^2 - 2\xi m_0^2\right] = 2\left(M\dot{M}^{(1)} - m_0^2\right)$, in the limit $\Lambda \to \infty$.

The one-loop correction to the coupling λ is given by

$$
\begin{aligned}
\lambda^{(1)} - \lambda_0 &= \frac{3\mathrm{i}\lambda_0^2}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{1}{\left(p^2 - 2\xi m_0^2\right)^2} \\
&= -\frac{3\lambda_0^2}{2}\frac{\Omega_2}{(2\pi)^2}\int_0^\infty \frac{q\,dq}{\left(q^2 + 2\xi m_0^2\right)^2} \\
&= -\frac{3\lambda_0^2}{16\pi\xi m_0^2},
\end{aligned}
\tag{35}
$$

where $3\lambda_0/2$ in the first line takes into account the symmetry factor and the three permutations of vanishing incoming momenta. The derivative of the latter result with respect to ξ indeed gives the above expression (32) for $\dot{\lambda}^{(1)}$.

The coupling β is generated by quantum fluctuations, and its one-loop expression is, taking into account the symmetry and permutation factors,

$$\begin{aligned}\beta^{(1)} &= \frac{6!}{8\times 6}(\mathrm{i}\lambda_0)^3\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{\left(p^2-2\xi m_0^2\right)^3}\\ &= 15\lambda_0^3\frac{\Omega_2}{(2\pi)^2}\int_0^\infty\frac{q\,dq}{\left(q^2+2\xi m_0^2\right)^3}\\ &= \frac{15\lambda_0^3}{32\pi\xi^2m_0^4}.\end{aligned}\tag{36}$$

The derivative of the latter result with respect to ξ indeed gives the above expression (32) for $\dot\beta^{(1)}$.

We checked here that our non-perturbative evolution equations (30) are consistent, at one loop, with usual Feynman graphs. This feature is a consequence of the fact that, in the framework of the gradient expansion (15), the evolution of U_ξ is independent of the evolution of V_ξ. Beyond one loop, the gradient expansion does not give the same results than the loop expansion, as it is based on an expansion in powers of the momentum.

3.4. Approximate analytical solution

We are interested here in approximate analytical solutions for the parameters λ and v_1.

An approximate solution for λ given in equations (30) can be obtained by keeping the bare values for the other parameters: $M\to\sqrt{2\xi}m_0$ and $\beta\to 0$, in which case the equation for λ reads

$$\frac{\dot\lambda}{\lambda^2}=\frac{3}{16\pi\xi^2m_0^2}.\tag{37}$$

We see that, as quantum fluctuations arise (ξ decreases), λ decreases ($\dot\lambda>0$), which was expected, as a scalar self-coupling is known to decrease in the IR. If we define the renormalized coupling $\lambda_R=\lambda(1)$, the solution of equation (37) can easily be found and reads

$$\lambda(\xi)=\lambda_R\left[1+\frac{3\lambda_R}{16\pi m_0^2}\left(\frac{1}{\xi}-1\right)\right]^{-1}.\tag{38}$$

In the spirit of the present functional method, the bare coupling λ_0 should be found in the limit $\xi\to\infty$, which leads to the following expression for the dressed coupling in terms of the bare coupling:

$$\lambda_R=\lambda_0\left(1+\frac{3\lambda_0}{16\pi m_0^2}\right)^{-1}.\tag{39}$$

Using a similar approximation in the evolution equation for the parameter v_1, i.e. $M\to\sqrt{2\xi}m_0$ and $v_2\to 0$, we find the following evolution for v_1:

$$\frac{\dot v_1}{v_1}=\frac{\lambda}{24\pi\xi^2m_0^2},\tag{40}$$

where λ is given by equation (38). The integration of this equation then gives

$$v_1(\xi)=3m_R^2\left[1+\frac{3\lambda_R}{16\pi m_0^2}\left(\frac{1}{\xi}-1\right)\right]^{-2/9},\tag{41}$$

where we define $3m_R^2 = v_1(1)$. As previously, a relation between the renormalized parameters λ_R and m_R^2 can be obtained, by taking the limit $\xi \to \infty$ in the previous equation, with $v_1 \to 3m_0^2$, such that

$$m_R^2 = m_0^2 \left(1 - \frac{3\lambda_R}{16\pi m_0^2}\right)^{2/9}. \tag{42}$$

From relation (39), we then obtain the following expression for m_R^2 in terms of the bare parameters only:

$$m_R^2 = m_0^2 \left(\frac{16\pi m_0^2}{3\lambda_0 + 16\pi m_0^2}\right)^{2/9}. \tag{43}$$

Equations (39), (43) consist in a ressumation in all orders in $\hbar$ and are derived in the limit of a highly localized topological defect, $m_0^2 \gg \lambda_0$.

4. Discussion

We now discuss the vanishing of the dressed term $\tanh(\zeta)Y_\xi(\phi)$ in the effective action, as a consequence of a discrete symmetry of the theory.

In this work, we considered the quantization of fluctuations above the kink Φ_{bg} given in equation (4), but the classical equation of motion (3) has actually two kink solutions centered on $z = 0$, which are $\pm\Phi_{bg}$. We now discuss the symmetry of the quantum theory under the transformation $\Phi_{bg} \to -\Phi_{bg}$, and we denote by an upper indice $^{(\pm)}$ the different quantities defined respectively on the backgrounds $\pm\Phi_{bg}$. We note here that the vacuum of the theory, the constant configuration $\Phi_0 = m_0\sqrt{6/\lambda_0}$, does not respect the symmetry $\Phi_0 \to -\Phi_0$, since this symmetry is spontaneously broken.

The bare action corresponding to the background $-\Phi_{bg}$ is

$$\begin{aligned} S_\xi^{(-)}[\phi] &= \int \mathrm{d}t\,\mathrm{d}z \left\{\frac{1}{2}\partial_\mu\tilde\Phi\partial^\mu\tilde\Phi - \xi m_0^2\tilde\Phi^2 - \frac{\lambda_0}{24}\tilde\Phi^4 + \frac{3}{2}m_0^2[1-\tanh^2(\zeta)]\tilde\Phi^2 + \frac{g_0}{6}\tanh(\zeta)\tilde\Phi^3\right\} \\ &= S_\xi^{(+)}[\psi], \end{aligned} \tag{44}$$

where $\psi(t, z) = \phi(t, -z)$. The source term can then be written as

$$\int \mathrm{d}t\,\mathrm{d}z\, j\phi = \int \mathrm{d}t\,\mathrm{d}z\, g\psi, \tag{45}$$

where $g(t, z) = j(t, -z)$, such that the partition function is

$$\begin{aligned} Z_\xi^{(-)}[j] &= \int \mathcal{D}[\phi] \exp\left\{\mathrm{i}S^{(-)}[\phi] + \mathrm{i}\int \mathrm{d}t\,\mathrm{d}z\, j\phi\right\} \\ &= \int \mathcal{D}[\psi] \exp\left\{\mathrm{i}S^{(+)}[\psi] + \mathrm{i}\int \mathrm{d}t\,\mathrm{d}z\, g\psi\right\} \\ &= Z_\xi^{(+)}[g]. \end{aligned} \tag{46}$$

The classical field corresponding to the background $-\Phi_{bg}$ is

$$\begin{aligned} \phi_c^{(-)}(t, z) &= \frac{\delta W_\xi^{(-)}}{\delta j(t, z)} = \int \mathrm{d}s\,\mathrm{d}y \frac{\delta W_\xi^{(+)}}{\delta g(s, y)} \frac{\delta g(s, y)}{\delta j(t, z)} \\ &= \int \mathrm{d}s\,\mathrm{d}y \frac{\delta W_\xi^{(+)}}{\delta g(s, y)} \delta(y + z)\delta(s - t) = \frac{\delta W_\xi^{(+)}}{\delta g(t, -z)} = \frac{\delta W_\xi^{(+)}}{\delta j(t, z)} \\ &= \phi_c^{(+)}(t, z), \end{aligned}$$

and is therefore independent of the sign of the background: $\phi_c^{(-)} = \phi_c^{(+)}$. When defining the Legendre transform Γ_ξ, we inverse the relation $j \to \phi_c$, such that the source is now a function of the background, and therefore $j^{(-)} = j^{(+)}$. The effective action is then

$$\begin{aligned}\Gamma_\xi^{(-)}[\phi_c] &= W_\xi[j^{(-)}] - \int \mathrm{d}t\,\mathrm{d}z\, j^{(-)}\phi_c \\ &= W_\xi[j^{(+)}] - \int \mathrm{d}t\,\mathrm{d}z\, j^{(+)}\phi_c \\ &= \Gamma_\xi^{(+)}[\phi_c].\end{aligned} \tag{47}$$

As a consequence, the effective action does not depend on the sign of the background, such that the dressed term $\tanh(\zeta)Y_\xi(\phi_c)$ in the effective action (15) must vanish, as it should satisfy $-Y_\xi = Y_\xi$. The corresponding term in the bare action does not survive quantization. It is interesting to note that the non-perturbative method presented here allows us to see the vanishing of the dressed potential Y_ξ, using equations (25), (26), which means that quantum fluctuations are strong enough to cancel the corresponding term present in the bare action. This could not be obtained within a perturbative approach, but only within a method using a self-consistent equation as equation (14).

The next step in this study consists in including fermions coupled to the scalar field fluctuating over the background kink. From the results obtained here, we can expect a usual Yukawa coupling $\phi\overline{\psi}\psi$ to be relevant to the problem, or more generally a coupling of the form $f(\phi)\overline{\psi}\psi$, without an explicit z-dependence. As explained in appendix B, the evolution equation for the effective action Γ_ξ with ξ is then obtained in the same way, with more involved calculations though, as the second derivative $\delta^{(2)}\Gamma$ is then a 3×3 matrix, with rows ϕ, $\overline{\psi}$, ψ, and the computation of the trace in equation (14) involves the inverse of this matrix. In higher dimensions, the method presented here is of course valid and can include any other matter field. Also, it can be extended to higher symmetries and deal with gauge fields. As far as supersymmetry is concerned, the use of the superfield formalism necessitates a modification of the evolution equation (14), which takes into account the chirality constraint of the superfields.

Finally, we emphasize the advantage of the present approach, in 1+1 dimensions, compared to the Wilsonian approach: we were able to generate non-perturbative flows without referring to a running cut-off, as no cut-off appears in the evolution for the dressed potentials, and the integration of the corresponding flows led us to cut-off-free relations between bare and dressed parameters.

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Appendix A. Computation of the trace

For the step-like configuration $\phi = \mathrm{sign}(z)\phi_0$, the second derivative of the effective action is

$$\frac{\delta^2\Gamma_\xi}{\delta\phi_1\delta\phi_2} = \{-\partial_\mu\partial^\mu - U_\xi''(\phi_0) + [1 - \tanh^2(\zeta)]V_\xi''(\phi_0) - |\tanh(\zeta)|Y_\xi''(\phi_0)\}\delta(t_1 - t_2)\delta(z_1 - z_2), \tag{A.1}$$

and therefore we need the Fourier transform of $1 - \tanh^2(\zeta)$ and $|\tanh(\zeta)|$. For this, we approximate the function $\tanh(\zeta)$ by ζ in the interval $[-1; 1]$ and by 0 outside this interval.

This approximation captures the essential features of the kink, and leads, for the Fourier transform of $1 - \tanh^2(\zeta)$, to

$$\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz}[1 - \tanh^2(\zeta)] \simeq \int_{-1/m_0}^{1/m_0} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz}(1 - \zeta^2)$$
$$= 4\frac{m_0^2}{k^3} \sin\left(\frac{k}{m_0}\right) - 4\frac{m_0}{k^2} \cos\left(\frac{k}{m_0}\right). \tag{A.2}$$

The same approximation leads, for the Fourier transform of $|\tanh(\zeta)|$, to

$$\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz}|\tanh(\zeta)| \simeq \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz} + \int_{-1/m_0}^{1/m_0} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz}(|\zeta| - 1)$$
$$= 2\pi\delta(k) + 2\frac{m_0}{k^2}\left[\cos\left(\frac{k}{m_0}\right) - 1\right]. \tag{A.3}$$

Since we are interested in the limit of a highly localized topological defect, we consider the situation where $m_0 \gg |k|$, which gives

$$\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz}[1 - \tanh^2(\zeta)] \simeq \frac{4}{3m_0} + \mathcal{O}\left(\frac{k^2}{m_0^3}\right)$$
$$\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}kz}|\tanh(\zeta)| \simeq 2\pi\delta(k) - \frac{1}{m_0} + \mathcal{O}\left(\frac{k^2}{m_0^3}\right).$$

The inverse $(\delta^2\Gamma)^{-1}$ is taken as

$$(A + B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \cdots, \tag{A.4}$$

where A is proportional to $\delta(\omega_1 + \omega_2)\delta(k_1 + k_2)$ and is thus diagonal and B is proportional to $\delta(\omega_1 + \omega_2)$ only and is thus off-diagonal in the space dimension. We then obtain, taking into account the first order in B in expansion (A.4),

$$\left(\frac{\delta^2\Gamma_\xi}{\delta\phi_1\delta\phi_2}\right)^{-1} \simeq \frac{2\pi\delta(\omega_1 + \omega_2)2\pi\delta(k_1 + k_2)}{\omega_1^2 - k_1^2 - U_\xi''(\phi_0) - Y_\xi''(\phi_0)}$$
$$- \frac{(3m_0)^{-1}[4V_\xi''(\phi_0) + 3Y_\xi''(\phi_0)]2\pi\delta(\omega_1 + \omega_2)}{\left[\omega_1^2 - k_1^2 - U_\xi''(\phi_0) - Y_\xi''(\phi_0)\right]\left[\omega_2^2 - k_2^2 - U_\xi''(\phi_0) - Y_\xi''(\phi_0)\right]}. \tag{A.5}$$

The term proportional to LT in the trace of equation (14) is

$$LT \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi} \frac{1}{\omega^2 - k^2 - U_\xi''(\phi_0) - Y_\xi''(\phi_0)} = -\mathrm{i}LT\frac{\Omega_2}{(2\pi)^2}\int_0^\Lambda \frac{q\,dq}{q^2 + U_\xi''(\phi_0) + Y_\xi''(\phi_0)}$$
$$= -LT\frac{\mathrm{i}}{4\pi}\ln\left(1 + \frac{\Lambda^2}{U_\xi''(\phi_0) + Y_\xi''(\phi_0)}\right), \tag{A.6}$$

where q is the Euclidean 2-momentum and $\Omega_2 = 2\pi$ is the solid angle in dimension 2.

The term proportional to T only in the trace of equation (14) is

$$-\frac{4V_\xi''(\phi_0) + 3Y_\xi''(\phi_0)}{3m_0}T\int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi} \frac{1}{(\omega^2 - k^2 - U_\xi''(\phi_0) - Y_\xi''(\phi_0))^2}$$
$$= -\frac{4V_\xi''(\phi_0) + 3Y_\xi''(\phi_0)}{3m_0}T\frac{\Omega_2}{(2\pi)^2}\int_0^\infty \frac{q\,dq}{(q^2 + U_\xi''(\phi_0) + Y_\xi''(\phi_0))^2}$$
$$= \frac{-\mathrm{i}T}{12\pi m_0}\frac{4V_\xi''(\phi_0) + 3Y_\xi''(\phi_0)}{U_\xi''(\phi_0) + Y_\xi''(\phi_0)}. \tag{A.7}$$

The left-hand side of equation (14) is, for the step-like configuration,

$$LT\left[m_0^2\phi_0^2 - \dot{U}_\xi(\phi_0) - \dot{Y}_\xi(\phi_0)\right] + \frac{T}{m_0}[2\dot{V}_\xi(\phi_0) + \ln 2\dot{Y}_\xi(\phi_0)], \tag{A.8}$$

and, together with equations (A.6), (A.7), we obtain for the evolution of the dressed potentials

$$\begin{aligned} \dot{U}_\xi(\phi_0) + \dot{Y}_\xi(\phi_0) &= m_0^2\phi_0^2 + \frac{m_0^2}{4\pi}\ln\left(1 + \frac{\Lambda^2}{U''_\xi(\phi_0) + Y''_\xi(\phi_0)}\right) \\ \dot{V}_\xi(\phi_0) + \frac{\ln 2}{2}\dot{Y}_\xi(\phi_0) &= -\frac{m_0^2}{24\pi}\frac{4V''_\xi(\phi_0) + 3Y''_\xi(\phi_0)}{U''_\xi(\phi_0) + Y''_\xi(\phi_0)}. \end{aligned} \tag{A.9}$$

Appendix B. Extension to a Yukawa interaction

We give here the main steps of the extension of the previous method to a Yukawa interaction in $d+1$ dimensions, where the kink expands in the extra dimension, with coordinate z. For $d \geqslant 4$, the theory is not renormalizable, and we consider it an effective theory, valid up to an energy scale Λ, which is our cut-off.

The bare action is, for massless fermions,

$$S_0 = \int \mathrm{d}^d x\, \mathrm{d}z \left\{ \mathrm{i}\overline{\Psi}\partial\!\!\!/\,\Psi + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \eta_0\Phi\overline{\Psi}\Psi - U_B(\Phi) \right\}, \tag{B.1}$$

where the scalar potential $U_B(\Phi)$ is given in equation (2). The fermion field having no expectation value, the kink configuration is the same as in equation (4), and the action to quantize is

$$\begin{aligned} S_\xi = \int \mathrm{d}^d x\, \mathrm{d}z \Bigg\{ & \mathrm{i}\overline{\Psi}\partial\!\!\!/\,\Psi - \eta_0\Phi_{\mathrm{bg}}(\zeta)\overline{\Psi}\Psi - \eta_0\tilde{\Phi}\overline{\Psi}\Psi + \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} - \xi m_0^2\tilde{\Phi}^2 - \frac{\lambda_0}{24}\tilde{\Phi}^4 \\ & + \frac{3}{2}m_0^2[1 - \tanh^2(\zeta)]\tilde{\Phi}^2 - \frac{g_0}{6}\tanh(\zeta)\tilde{\Phi}^3 \Bigg\}, \end{aligned} \tag{B.2}$$

where $\tilde{\Phi}$ represent the fluctuations above the classical kink. In the previous expression, the z-dependent mass term $\eta_0\Phi_{\mathrm{bg}}(\zeta)\overline{\Psi}\Psi$ for the fermion is responsible for the fermion localization on the brane $z = 0$, as discussed in [1].

The partition function, functional of the sources $j, \eta, \overline{\eta}$, is

$$\begin{aligned} Z_\xi &= \int \mathcal{D}[\tilde{\Phi}, \Psi, \overline{\Psi}] \exp\{\mathrm{i}S_\xi[\tilde{\Phi}, \Psi, \overline{\Psi}] + \mathrm{i}\int \mathrm{d}^d x\, \mathrm{d}z (j\tilde{\Phi} + \overline{\eta}\Psi + \overline{\Psi}\eta)\} \\ &= \exp(\mathrm{i}W_\xi[j, \eta, \overline{\eta}]), \end{aligned} \tag{B.3}$$

from which the classical fields $(\phi, \psi, \overline{\psi})$ are defined:

$$\frac{\delta W_\xi}{\delta j} = \phi \qquad \frac{\delta W_\xi}{\delta \overline{\eta}} = \psi \qquad \frac{\delta W_\xi}{\delta \eta} = -\overline{\psi}. \tag{B.4}$$

The proper graphs generator functional of the classical fields $\phi, \overline{\psi}, \psi$ is defined as the Legendre transform of W, after inverting the relations $(j, \eta, \overline{\eta}) \to (\phi_\xi, \psi_\xi, \overline{\psi}_\xi)$ to $(\phi, \psi, \overline{\psi}) \to (j_\xi, \eta_\xi, \overline{\eta}_\xi)$:

$$\Gamma_\xi[\phi, \psi, \overline{\psi}] = W_\xi[j, \eta, \overline{\eta}] - \int \mathrm{d}^d x\, \mathrm{d}z (j_\xi\phi + \overline{\eta}_\xi\psi + \overline{\psi}\eta_\xi), \tag{B.5}$$

and its functional derivatives are

$$\frac{\delta\Gamma_\xi}{\delta\phi} = -j_\xi \qquad \frac{\delta\Gamma_\xi}{\delta\overline{\psi}} = -\eta_\xi \qquad \frac{\delta\Gamma_\xi}{\delta\psi} = \overline{\eta}_\xi. \tag{B.6}$$

The evolution equation for Γ_ξ with ξ is derived in the same way as was done in 1+1 dimensions and reads

$$\dot{\Gamma}_\xi + m_0^2 \int \mathrm{d}^d x\, \mathrm{d}z\, \phi^2 = -\mathrm{i} m_0^2 \mathrm{Tr}\left\{(\delta^2\Gamma_\xi)^{-1}_{\phi\phi}\right\}, \tag{B.7}$$

but this time $(\delta^2\Gamma_\xi)^{-1}_{\phi\phi}$ is the $\phi\phi$ component of the inverse of the matrix

$$\delta^2\Gamma_\xi = \begin{pmatrix} \frac{\delta^2\Gamma}{\delta\overline{\psi}\delta\psi} & \frac{\delta^2\Gamma}{\delta\overline{\psi}\delta\overline{\psi}} & \frac{\delta^2\Gamma}{\delta\overline{\psi}\delta\phi} \\ \frac{\delta^2\Gamma}{\delta\psi\delta\psi} & \frac{\delta^2\Gamma}{\delta\psi\delta\overline{\psi}} & \frac{\delta^2\Gamma}{\delta\psi\delta\phi} \\ \frac{\delta^2\Gamma}{\delta\phi\delta\psi} & \frac{\delta^2\Gamma}{\delta\phi\delta\overline{\psi}} & \frac{\delta^2\Gamma}{\delta\phi\delta\phi} \end{pmatrix}. \tag{B.8}$$

Note that the components $\delta^2\Gamma_{\overline{\psi}\overline{\psi}}$ and $\delta^2\Gamma_{\psi\psi}$ do not vanish in general, as quantum fluctuations generate 4-fermion interactions and higher powers of $\overline{\psi}\psi$. Also, compared to the 1+1 dimensional model, the trace in equation (B.7) contains divergences, such that the cut-off Λ will appear in the final equations.

In order to take into account fermion localization, and the symmetries of the system, we propose here the following gradient expansion, where we disregard higher order fermion interactions and wavefunction renormalization:

$$\begin{aligned} \Gamma_\xi = \int \mathrm{d}^d x\, \mathrm{d}z \Bigg\{ & \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U^1_\xi(\phi)\overline{\psi}\psi - U^2_\xi(\phi) \\ & + [1-\tanh^2(\zeta)]\left[\mathrm{i}\overline{\psi}\partial\!\!\!/\,\psi + V^1_\xi(\phi)\overline{\psi}\psi + V^2_\xi(\phi)\right]\Bigg\}. \end{aligned} \tag{B.9}$$

In the previous expression, fermion localization is implemented via the ζ-dependence fermion kinetic term: away from the brane, for $\zeta \neq 0$, the fermion propagation is exponentially damped. The consistency of this ansatz for the functional dependence of Γ_ξ has to be checked when computing the trace in equation (B.7) and following the evolution of Γ_ξ with ξ, which leads to the evolution of the scalar potentials $U^{1,2}_\xi(\phi)$ and $V^{1,2}_\xi(\phi)$.

An interesting study is then to look for a possible mass generated dynamically, on the brane, for the would-be massless fermion. This can be investigated using the present method, since it is based on a self-consistent equation, as was already done in the Kaluza–Klein framework [9]. In the present context, the fermion dynamical mass, if there is one, is $m_{\mathrm{dyn}} = U^1_\xi(0) - V^1_\xi(0)$.

References

[1] Jackiw R and Rebbi C 1976 *Phys. Rev.* D **13** 3398
Rubakov V A and Shaposhnikov M E 1983 *Phys. Lett.* B **125** 136
Callan C G and Harvey J A 1985 *Nucl. Phys.* B **250** 427

[2] Kaplan D B 1992 *Phys. Lett.* B **288** 342 (*Preprint* hep-lat/9206013)

[3] Ambjorn J and Farakos K 1992 *Phys. Lett.* B **294** 248 (*Preprint* hep-lat/9207020)
Ambjorn J, Farakos K, Hands S, Koutsoumbas G and Thorleifsson G 1994 *Nucl. Phys.* B **425** 39 (*Preprint* hep-lat/9401028)

[4] Jackiw R 1977 *Rev. Mod. Phys.* **49** 681

[5] Kapustnikov A A, Pashnev A and Pichugin A 1997 *Phys. Rev.* D **55** 2257 (*Preprint* hep-th/9608124)
Zloshchastiev K G 1998 *J. Phys. A: Math. Gen.* **31** 6081 (*Preprint* hep-th/9708018)

Izquierdo A Alonso, Garcia Fuertes W, Gonzalez Leon M A and Mateos Guilarte J 2004 *Nucl. Phys.* B **681** 163 (*Preprint* hep-th/0304125)
Mussardo G, Riva V, Sotkov G and Delfino G 2006 *Nucl. Phys.* B **736** 259 (*Preprint* hep-th/0510102)
Sanchez A, Vazquez L and Konotop V V 1991 *Phys. Rev.* A **44** 1086
Dorey N, Mattis M P and Hughes J 1994 *Phys. Rev.* D **49** 3598 (*Preprint* hep-th/9309018)
[6] Garrahan J P, Kruczenski L M, Schat C L, Bes D R and Scoccola N N 1995 *Phys. Rev.* D **51** 2950 (*Preprint* hep-th/9412146)
[7] Berges J, Tetradis N and Wetterich C 2002 *Phys. Rep.* **363** 223 (*Preprint* hep-ph/0005122)
Morris T R 2000 *Nucl. Phys.* B **573** 97 (*Preprint* hep-th/9910058)
Litim D F and Pawlowski J M 2002 *Phys. Rev.* D **65** 081701 (*Preprint* hep-th/0111191)
[8] Alexandre J and Polonyi J 2001 *Ann. Phys.* **288** 37 (*Preprint* hep-th/0010128)
Alexandre J, Polonyi J and Sailer K 2002 *Phys. Lett.* B **531** 316 (*Preprint* hep-th/0111152)
[9] Alexandre J and Farakos K 2006 *New J. Phys.* **8** 198 (*Preprint* hep-ph/0603016)
[10] Alexandre J, Ellis J and Mavromatos N E 2006 *J. High Energy Phys.* JHEP12(2006)071 (*Preprint* hep-th/0610072)
[11] Polonyi J and Sailer K 2005 *Phys. Rev.* D **71** 025010 (*Preprint* hep-th/0410271)